UDC 539.375

INFLUENCE OF DEFORMATIONAL ANISOTROPY ON THE STATE NEAR THE CRACK TIP

PMM Vol. 41, № 5, 1977, pp. 943 - 948.

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(Received October 11, 1977)

Stress-strain near a crack tip generated by longitudinal shear is considered at the initial stage of its development, under the conditions of a nonsteady, quasi-static regime. The defining relations of the slip theory are adopted [1]. The asymptotic solution of the problem constructed in the first approximation using the method of a small parameter, is compared with a similar solution obtained from the theory of flow with isotropic hardening [2]. The influence of weak anisotropy on the order of the singularity and the peripheral distribution of stress rates are assessed.

1. The relations of the theory of plasticity based on the concept of slip [1] predict a sharp deformational anisotropy of the medium, especially in the case when additional loading occurs from the state arrived at by a preliminary active process with a well defined direction of development. This specifies the part played by the theory in estimating the influence of the accumulated anisotropy on the structure of the solution near the end of a crack developing in an elastoplastic material.

In the case of antiplane deformation, the defining relations of the slip theory admit the formulation similar to the two-dimensional version which was studied in [3]. Let us refer the homogeneous state of a small element of the medium to the Cartesian x, y, z coordinate system and denote by $\mathbf{n}$ the vector normal to the running slip plane inclined to the x-axis at the angle ω. Let shear stresses τ_{xz} and τ_{yz} act on the element so that we have $\tau_{nz} = \tau_{xz}\cos\omega + \tau_{yz}\sin\omega$ on the slip plane. Then a local shear of magnitude $\gamma_{nz} = F(\tau_{nz})$ will appear in the slip system. According to the basic concept of the slip theory, the plastic deformation can be found by performing a summation (the subscript z is omitted)

$$\gamma^p = \int F(\tau_n)(\mathbf{e}_1\cos\lambda + \mathbf{e}_2\sin\lambda)\,d\lambda \tag{1.1.}$$

$$\tau_x = \tau\cos\beta,\ \ \tau_y = \tau\sin\beta,\ \ \lambda = \omega - \beta,\ \ \tau_n = \tau\cos\lambda \geqslant \tau_0$$

$$\cos\lambda_0 = \tau_0/\tau,\ \ \mathbf{e}_1 = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta,\ \mathbf{e}_2 = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta$$

Here $\mathbf{i}$ and $\mathbf{j}$ denote the unit vectors along the axes of τ_x and τ_y; β is the loading angle and τ_0 denotes the reaction opposing the shear in the slip system. The integration is carried out over the region $|\lambda| \leqslant \lambda_0$, since $F = 0$ for $\tau < \tau_0$.

The relation (1.1) yields the following expression for the additional loading:

$$\delta\gamma^p = \int_a^b F'(\tau_n)\,\delta\tau\cos(\lambda - \alpha)(\mathbf{e}_1\cos\lambda + \mathbf{e}_2\sin\lambda)\,d\lambda \tag{1.2}$$

$$\delta\tau_n = \delta\tau\cos(\lambda - \alpha),\quad \delta\boldsymbol{\tau} = \delta\tau(\cos(\beta + \alpha)\,\mathbf{i} + \sin(\beta + \alpha)\,\mathbf{j})$$

For an active process, the limits of integration in (1.2) are determined from the conditions $|\lambda| \leqslant \lambda_0$, $\cos(\lambda - \alpha) \geqslant 0$. For $\alpha \geqslant 0$ they have the form

$$a = -\lambda_0, \quad b = \lambda_0 \quad (0 \leqslant \alpha \leqslant \pi / 2 - \lambda_0) \qquad (1.3)$$
$$a = \alpha - \pi / 2, \quad b = \lambda_0 \quad (\pi / 2 - \lambda_0 < \alpha \leqslant \pi / 2 + \lambda_0)$$

and correspond to the regions of complete and incomplete loading. When $\pi / 2 + \lambda_0 < \alpha \leqslant \pi$, the additional plastic deformations in (1.2) vanish.

In order to simplify the relationships (1.1) and (1.2) we shall assume that for $\tau_n \geqslant \tau_0$ the local slip mechanism is determined by the simplest possible relation $F(\tau_n) = C(\tau_n - \tau_0)$. Performing the integration in (1.1) and adding the elastic deformations $\gamma_i'' = \tau_i / G$, we see that the relation governing the deviator intensities $\tau(\gamma) \in C_1(0, \infty)$, and

$$\tau = G\gamma \ (\gamma \leqslant \gamma_0), \quad \tau' \to \frac{G}{1+\mu} \quad \left(\gamma \to \infty, \mu = \frac{G}{g}, g = \frac{2}{\pi C}\right) \qquad (1.4)$$

We note that the inclusion of the higher powers in the expansion for $F(\tau_n)$ also leads to the case of power law hardening for the macrodiagram $\tau(\gamma)$. The hardening index however is found to differ from unity when $\gamma \to \infty$. The latter fact can also be established using the method given in [4].

Integrating (1.2) and taking (1.3) into account, we obtain the additional plastic deformations

$$\delta\gamma_x^p = \frac{\delta\tau}{\pi g} \times \begin{cases} \sin 2\lambda_0 \cos(\beta - \alpha) + 2\lambda_0 \cos(\beta + \alpha), \ 0 \leqslant \alpha \leqslant \pi/2 - \lambda_0 \\ \cos(\alpha - \lambda_0)\sin(\beta + \lambda_0) + (\lambda_0 - \alpha + \pi/2) \times \\ \quad \cos(\beta + \alpha), \ |\alpha - \pi/2| \leqslant \lambda_0 \\ 0, \ \pi/2 + \lambda_0 < \alpha \leqslant \pi \end{cases} \qquad (1.5)$$

$$\delta\gamma_y^p = \frac{\pi\delta\tau}{g} \times$$
$$\begin{cases} \sin 2\lambda_0 \sin(\beta - \alpha) + 2\lambda_0 \sin(\beta + \alpha), \ 0 \leqslant \alpha \leqslant \pi/2 - \lambda_0 \\ -\cos(\alpha - \lambda_0)\cos(\beta + \lambda_0) + (\lambda_0 - \alpha + \pi/2)\sin(\beta + \alpha), \ |\alpha - \pi/2| \leqslant \lambda_0 \\ 0, \ \pi/2 + \lambda_0 < \alpha \leqslant \pi \end{cases}$$

We see from (1.5) that the relation between the stress and deformation increments is differentially linear in the zone of complete loading, in constrast to the analogous relation for the zone of incomplete loading where the nonlinearity is caused by the appearance in the equations of the explicit quantity $\alpha = \operatorname{arctg}(\delta\tau_x / \delta\tau_y) - \beta$ and complicates, in general, their use.

2. Let an elastoplastic body with a crack of cut along the ray $x \leqslant 0, y = 0$ be subjected to a longitudinal shear in the z-direction. By virtue of the singular charactor of the stresses we find that in a sufficiently small neighborhood of the tip, the state preceding the onset of crack propagation and corresponding to some value of the stress intensity factor K_0, is attained under the loading which is nearly proportional. From the linearity of the asymptotics of the diagram it follows from (1.4) that the principal term of the stress field has the form [2] (r, ϑ are the polar coordinates)

$$\tau_y^\circ = \frac{K_0}{r^{1/2}} \cos\frac{\vartheta}{2}, \quad \tau_x^\circ = -\frac{K_0}{r^{1/2}} \sin\frac{\vartheta}{2} \qquad (2.1)$$

Using (1.1) and (2.1) we find that $\beta = (\pi + \vartheta) / 2$.

Let us assume that when K increases, the crack length increases by δl and a redistribution of τ_i and γ_i takes place in accordance with Eqs. (1.5) of the theory.

Using (1.1) and (2.1) we find that $\tau / \tau_0 \to \infty$, $\lambda_0 \to \pi / 2$, when $r \to 0$, and $\delta l \to 0$ ($\delta l / r = o(1)$), therefore the zones of complete loading and unloading degenerate into the lines $\alpha = 0$ and π respectively, while the zone of incomplete loading is realized for all angles $|\alpha| < \pi$ of additional loading. Neglecting in (1.4) terms of higher order smallness compared with $\delta\tau$ and passing to the polar coordinate system we obtain, after some manipulations, the following nonlinear equations:

$$G\delta\gamma_r = \delta\tau_r + \mu f_1(\alpha, \vartheta)\,\delta\tau, \quad G\delta\gamma_\vartheta = \delta\tau_\vartheta + \mu f_2(\alpha, \vartheta)\,\delta\tau \tag{2.2}$$

$$f_1(\alpha, \vartheta) = \left[\frac{1}{\pi}\sin\alpha\sin\frac{\vartheta}{2} + \left(1 - \frac{\alpha}{\pi}\right)\sin\left(\frac{\vartheta}{2} - \alpha\right)\right]$$

$$f_2(\alpha, \vartheta) = \left[\frac{1}{\pi}\sin\alpha\cos\frac{\vartheta}{2} + \left(1 - \frac{\alpha}{\pi}\right)\cos\left(\frac{\vartheta}{2} - \alpha\right)\right]$$

$$(\delta\gamma_r = \partial(\delta w) / \partial r, \quad \delta\gamma_\vartheta = \partial(\delta w) / r\partial\vartheta)$$

The relations (2.2) predict an active process under any direction of additional loading, since the angle at the tip of the conical singularity of the loading surface is infinitely small. Indeed, from (1.2) (or (2.2)) follows

$$|\delta\gamma^p| = (\delta\tau / g\pi)\,[\sin^2\alpha + (\pi - \alpha)^2 + (\pi - \alpha)\sin 2\alpha]^{1/2} \geqslant 0$$
$$0 \leqslant \alpha \leqslant \pi$$

We note that the specific work done by the plastic deformations $\delta A_p = (\tau\delta\tau / g\pi)\,[\sin\alpha + (\pi - \alpha)]$ is also nonnegative. For this reason the defining equations (2.2) represent, at the same time, a variant of the analytic theory of plasticity introduced in [4] along a different route and also leading to equations of the differentially-nonlinear type.

Passing in (2.2) to the rates of change of the parameters and introducing the stress rate function $\Phi(r, \vartheta)$ ($\tau_r' = \partial\Phi / r\partial\vartheta$, $\tau_\vartheta' = -\partial\Phi / \partial r$; where a prime denotes differentiation with respect to the dimensionless load parameter, we substitute (2.2) into the compatibility equation

$$r\Delta\Phi = \mu\left[\tau' f_2(\alpha, \vartheta) + r\frac{\partial}{\partial r}(\tau' f_2(\alpha, \vartheta)) - \frac{\partial}{\partial\vartheta}(\tau' f_1(\alpha, \vartheta))\right] \tag{2.3}$$

(Δ is the Laplace operator and the functions $f_1(\alpha, \vartheta)$ and $f_2(\alpha, \vartheta)$ are defined in (2.2))

The boundary conditions for the case of stress-free edges are

$$\Phi_\vartheta'(r, 0) = 0, \quad \Phi(r, \pi) = 0 \tag{2.4}$$

We seek the solution of the boundary value problem (2.3), (2.4), using the method of a small parameter. In the zeroth approximation, we set $\mu = 0$, $\Phi_0 = r^m\Psi_0(\vartheta)$, $-1/2 \leqslant m \leqslant 0$. Taking into account the fact that $\Psi_0(\vartheta)$ is even, from (2.3) we obtain

$$\Psi_0'' + m^2\Psi_0 = 0, \quad \Psi_0 = \cos m\vartheta, \quad \alpha = n\vartheta \quad (n = 1/2 - m) \tag{2.5}$$

The solution (2.5) contains an undefined multiplier from the normalizing condition $\Psi_0(0) = 1$. Restricting ourselves to the case of nonlinearity $\mu = o(1)$, we substitute Ψ_0 and α from (2.5) into the right-hand side of (2.3).

Denoting the first approximation by $\Phi_1(r, \vartheta)$ we obtain, after the relevant manipulations, an equation for Ψ_1 and the boundary conditions:

$$\Psi_1'' + m^2\Psi_1 = \frac{\mu m n}{\pi}\left[\sin\left(n + \frac{1}{2}\right)\vartheta - \sin m\vartheta\right] \quad (\Phi_1 = r^m\Psi_1(\vartheta)) \qquad (2.6)$$

$$\Psi_1(0) = 1,\ \Psi_1'(0) = 0,\quad \Psi_1(\pi) = 0$$

The solution of (2.6) has the form

$$\Psi_1(\vartheta) = \cos m\vartheta - \frac{\mu n}{\pi}\left[\frac{1}{2} + \frac{m(1-m)}{2m-1}\right]\sin m\vartheta + \frac{\mu n}{\pi}\left[\frac{\vartheta}{2}\cos m\vartheta + \frac{m}{2m-1}\sin\left(n + \frac{1}{2}\right)\vartheta\right]$$

and the characteristic equation for m is

$$\cos m\pi - \frac{\mu(^1/_2 - m)}{m\pi}\left[\frac{1}{2} + \frac{m(1-m)}{2m-1}\right]\sin m\pi + \frac{\mu(^1/_2 - m)}{\pi}\left[\frac{\pi}{2}\cos m\pi + \frac{m}{2m-1}\sin(1-m)\pi\right] = 0 \qquad (2.7)$$

When $\mu = 0$ (Hook's body), (2.5) and (2.7) together yield $m = -^1/_2$, $\Psi_1 = \Psi_2$ and the solution (2.5) becomes exact.

Below we give the relation $m(\mu)$ obtained from (2.7) by numerical methods and presented for $(m_C(\mu))$

μ	0	0.05	0.1	0.2	0.5	1	10
$-m_C \cdot 10^3$	500	488	478	461	429	400	—
$-m_T \cdot 10^3$	500	492	485	472	439	413	234

For small μ we set in (2.7) $m = -^1/_2 + \varepsilon(\mu)$ $(\varepsilon = o(1))$, and this yields

$$\varepsilon(\mu) = 2\mu / \pi^2,\ m = -^1/_2 + 2\mu / \pi^2$$

The distribution of the stress rates calculated in accordance with (2.6), (2.7) is shown in Fig. 1 for $\mu = 0.1$ (curves *1* and *2* for T_r and T_ϑ respectively) and for $\mu = 0.5$ (curves *5* and *6*). The case of $\mu = 0$ is given for comparison using a dashed line. Here $T_i(\vartheta) = \tau_i' / r^{m-1}$, $i = r, \vartheta$.

3. Let us now consider the same problem within the framework of the isotropic flow theory with linear hardening

$$G\gamma_r' = \tau_r' + \mu\tau_0^{-1}\tau'\tau_r^\circ,\quad G\gamma_\vartheta' = \tau_\vartheta' + \mu\tau_0^{-1}\tau'\tau_\vartheta^\circ \qquad (3.1)$$

$$\tau'\tau_0 = \tau_r^\circ\tau_r' + \tau_\vartheta^\circ\tau_\vartheta',\quad \tau_0^2 = \tau_i^\circ\tau_i^\circ = f(q),\quad dq = (^1/_2\, d\gamma_i^p d\gamma_i^p)^{1/2}.$$

The quantities accompanied by the superscript $^\circ$ refer to the state of limiting equilibrium of the crack (2.1).

To find the principal term $\Phi(r, \vartheta) = r^m\Psi(\vartheta)$ of the asymptotic expansion of the Airy function, we have the following problem for the eigenvalues [2]:

$$\Psi''\left(1 + \mu\sin^2\frac{\vartheta}{2}\right) - \left(m - \frac{1}{2}\right)\mu\sin\vartheta\,\Psi' + \Psi m\left[(m-1)\times\left(1 + \mu\cos^2\frac{\vartheta}{2}\right) + 1 + \frac{\mu}{2}\right] = 0 \quad (0 \leqslant \vartheta \leqslant \vartheta_1);\ \Psi'(0) = 0,\ \Psi(0) = 1 \qquad (3.2)$$

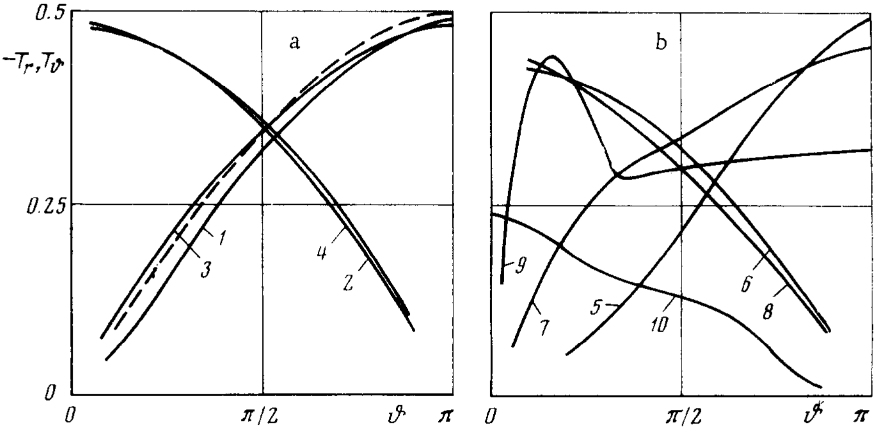
$$\Psi'' + m^2\Psi = 0 \quad (\vartheta_1 < \vartheta \leqslant \pi);\quad \Psi(\pi) = 0 \qquad (3.3)$$

$$[\Psi] = [\Psi'] = 0,\quad (\vartheta = \vartheta_1)$$

$$\Psi'(\vartheta_1)\sin\frac{\vartheta_1}{2} - m\Psi(\vartheta_1)\cos\frac{\vartheta_1}{2} = 0 \qquad (3.4)$$

The equations (3.2) -(3.4) take into account the appearance of the domains of elastic unloading in the regions adjacent to the crack edges, and the symmetry of the solution about the x -axis. It follows that the condition of neutral loading (3.4) must hold on the ray $\vartheta = \vartheta_1$

Fig. 1



Integrating (3.3) over the domain of unloading $\vartheta_1 \leqslant \vartheta \leqslant \pi$ and satisfying the boundary conditions from (3.3) and (3. 4) we find, that the problem has a nontrivial solution

$$\Psi = A \sin m(\pi - \vartheta) / \sin m\pi \quad (\vartheta_1 \leqslant \vartheta \leqslant \pi) \qquad (3.5)$$

under the condition that $\sin[(m - {}^1/_2)\vartheta_1 - m\pi] = 0$. The latter defines the value of ϑ_1 in the function of the singularity index [2]

$$\vartheta_1 = m\pi / (m - {}^1/_2) \quad (-{}^1/_2 \leqslant m \leqslant 0) \qquad (3.6)$$

Using the fact that the stress rates are continuous and the relation (3.5) we find that (3.4) yields

$$m\Psi(\vartheta_1)\cos\frac{m\pi}{1-2m} + \Psi'(\vartheta_1)\sin\frac{m\pi}{1-2m} = 0$$

The condition (3.7) closes the boundary value problem (3.2) and is used to determine the relation $m = m(\mu)$. The second condition of (3.2) is used here for normalizing.

The above boundary value problem was solved by numerical methods. The running iteration included integrating the Cauchy problem (3.2) on $[0, \vartheta_1]$, estimation of discrepancy for (3.7), and derivation of a more accurate value of m, by the method of dividing the segment in half. The condition of additional loading $\tau' \geqslant 0$ $(0 \leqslant \vartheta \leqslant \vartheta_1)$ and unloading $\tau' < 0$ $(\vartheta_1 < \vartheta \leqslant \pi)$ were checked for the solution obtained.

The relation $m_T(\mu)$ obtained is tabulated above, while the functions $T_r(\vartheta)$ (curves *3* and *7*) and $T_\vartheta(\vartheta)$ (curves *4* and *8*) are shown in Fig. 1 (curves *3* and *4* for $\mu = 0.1$, curves *7* and *8* for $\mu = 0.5$). The case of weak hardening $\mu = 10$ is depicted by the curves *9* (T_r) and *10* (T_ϑ).

Comparing the solutions of Sect. 2 and 3 we find, that there is little difference in the values of m and in the distributions of the component T_ϑ in the peripheral direction (not more than 1.5 and 3% for $\mu = 0.1$, 2.7 and 8% for $\mu = 0.5$). This implies that during the initial stage of crack development the influence of anisotropy on the level of concentration is not particularly significant. At the same time, the noticeable difference in the value of T_r (and $T = (T_i T_i)^{1/2}$) indicates the growing lack of overlap between the forms of the regions in which the deformations are considerable. For the medium (1.1), (1.2) the region of unrestricted plastic flow has, as we would expect, more fluid boundaries and its characteristic size exceeds that of the isotropy case (3.1). The deviations indicated have systematic character and it can apparently be assumed that the deviation of the loading trajectory in the elements near the tip from proportionality which increase with the growth of the crack, leads to an increase in the above mentioned effects.

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Translated by L. K.